VI. Conclusions

A periodic structure consisting of a linear array of conducting disks has been investigated, both theoretically and experimentally. The resonator experiments have established the fact that nonradiating surface waves propagate along the structure. These are slow waves, that is, their phase velocity is less than the phase velocity of an electromagnetic wave in an unbounded medium.

The propagation constants for the "dipole mode" have been experimentally determined for structures of different dimensions. By using the radiating aperture approach, the beamwidth has been calculated when these periodic structures are used as end-fire antennas.

Exact expressions for the field have been set up. Subject to certain approximations, a secular equation has been derived from which the propagation constants may be calculated. Theoretical values are compared with experiments.

The purity of the "dipole mode" on the disk structure has been verified by means of perturbation experiments. The disk line has thus been found suitable for use as an end-fire antenna, provided the "dipole mode" is efficiently excited, without a great amount of direct radiation from the feeding end.

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The Sinusoidal Variation of Dissipation Along Uniform Waveguides*

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Summary—**Standing waves in a waveguide with dielectric and/or metal wall losses generally give rise to expressions for power dissipation per unit length containing a term which is a sinusoidal function of the distance along the waveguide. In the present paper this phenomenon is explained and expressions for the dissipation are derived. The development is carried out for TE and TM modes in a uniform dielectric filled waveguide of arbitrary cross section, and then again from the standpoint of transmission line theory. The practical implications of the results are discussed.**

Introduction

When a standing wave exists in a lossy waveguide, the dissipation of power per unit length as a function of distance can generally be described by the sum of two terms, one of which is sinusoidal. While this fact has occasionally been recognized in one form or another,[1–3] it is commonly overlooked in practice, and no detailed treatment for the single mode case has, to the author's knowledge, been available. One can see intuitively that wall losses are greatest at the points of maximum magnetic field and dielectric losses at points of maximum electric field. Power attenuation in each traveling wave is therefore not proportional to $e^{\pm 2\alpha z}$ as is often assumed, and the improper use of such attenuation factors can lead to serious errors in the calculation of dissipation. This fact is not generally appreciated, although it is well known that in a dissipative guide it is not possible to speak of net power flow in terms of P_{inc} and P_{ref}, since these quantities are not well defined.

The purpose of this paper is to provide a firm basis for the accurate calculation of the distribution of dissipation along the direction of propagation and to point out the areas where calculations of this type are indicated.

Fig. 1 shows a section of uniform dielectric filled waveguide of arbitrary cross section with walls of surface resistivity r and filled with a dielectric material of conductivity σ. The dissipation in this waveguide will be found separately for the TE and TM mode cases. The analysis will be carried out in terms of normalized mode functions[4] (which, it is recognized,

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[3] R. W. P. King, "Transmission Line Theory," McGraw-Hill Book Co., Inc., New York, N. Y., p. 251; 1955.

[4] N. Marcuvitz, "Waveguide Handbook," McGraw-Hill Book Co., Inc., New York, N. Y., M.I.T. Rad. Lab. Ser., vol. 10; 1951.

are only approximate when the boundaries are not truly lossless). A self-contained alternate discussion deals with the problem from the transmission line point of view.

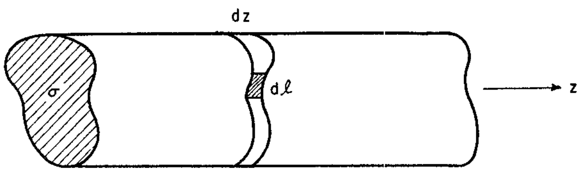

Fig. 1—Uniform dielectric filled waveguide.

TE Mode

In terms of the normalized electric field mode functions, the incident and reflected electric fields may be written[4] as

$$\boldsymbol{E}_{\substack{i\\r}}(r) = V_{\substack{i\\r}}(z)\boldsymbol{e}(\varrho) \tag{1}$$

where the subscripts i and r denote incident and reflected, respectively. Applying Maxwell's second equation, one has

$$\boldsymbol{H}_{\substack{i\\r}} = -\frac{1}{j\omega\mu}\nabla\times[V_{\substack{i\\r}}(z)\boldsymbol{e}(\varrho)] \tag{2}$$

which can be expanded as

$$\boldsymbol{H}_{\substack{i\\r}} = -\frac{1}{j\omega\mu}[\nabla V_{\substack{i\\r}}(z)\times\boldsymbol{e}(\varrho) + V_{\substack{i\\r}}(z)\nabla\times\boldsymbol{e}(\varrho)]. \tag{3}$$

In accordance with the small loss assumption, voltage changes over small distances depend almost entirely on $e^{\pm j\beta z}$ so that one can approximate

$$\nabla V_{\substack{i\\r}}(z) \approx \mp j\beta V_{\substack{i\\r}}(z)\boldsymbol{z}_0, \tag{4}$$

where the minus sign corresponds to the incident wave, and the plus sign to the reflected wave. Therefore,

$$\boldsymbol{H}_{\substack{i\\r}} = V_{\substack{i\\r}}(z)\left[\pm\frac{\beta}{\omega\mu}\boldsymbol{z}_0\times\boldsymbol{e}(\varrho) + \frac{j}{\omega\mu}\nabla\times\boldsymbol{e}(\varrho)\right]. \tag{5}$$

Referring to Fig. 1, the power dissipated in a volume filling the cross section and of differential length dz, which will be denoted by $-dP$, is given by

$$-dP = dz\left[\oint|J_w|^2 r\,dl + \int_s \frac{1}{\sigma}|J_d|^2 ds\right] \tag{6}$$

where J_w is the current density on the waveguide wall, and J_d the current density in the dielectric. Defining $D(z)$ as the dissipation per unit length at point z, one has

$$D(z) = -\frac{dP}{dz} = \oint|J_w|^2 r\,dl + \int_s \frac{1}{\sigma}|J_d|^2\,ds. \tag{7}$$

Since $J_d = \sigma\boldsymbol{E}$ and, under the small loss assumption, $|\boldsymbol{H}| = |J_w|$, (7) may also be written in the form

$$D(z) = \left[\oint|\boldsymbol{H}_i + \boldsymbol{H}_r|^2 r\,dl + \int_s \sigma|\boldsymbol{E}_i + \boldsymbol{E}_r|^2 ds\right], \tag{8}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are recognized to be phasors as well as vectors. Eq. (8) can therefore be expanded in terms of its real and imaginary parts:

$$D(z) = \oint[|\mathrm{Re}\,\boldsymbol{H}_i + \mathrm{Re}\,\boldsymbol{H}_r|^2 + |\mathrm{Im}\,\boldsymbol{H}_i + \mathrm{Im}\,\boldsymbol{H}_r|^2]r\,dl$$
$$+\int_s \sigma[|\mathrm{Re}\,\boldsymbol{E}_i + \mathrm{Re}\,\boldsymbol{E}_r|^2 + |\mathrm{Im}\,\boldsymbol{E}_i + \mathrm{Im}\,\boldsymbol{E}_r|^2]ds. \tag{9}$$

When the zero of the z scale is, for convenience, chosen at a voltage maximum, $V_i(z)$ and $V_r(z)$ may be written

$$V_i(z) = V_1(z)e^{-j\beta z}$$
$$V_r(z) = V_2(z)e^{j\beta z}, \tag{10}$$

where $V_1(z)$ and $V_2(z)$ are real positive amplitude factors. Substitution of (10) into (1) and (5) results in

$$(\mathrm{Re}\,\boldsymbol{H}_i + \mathrm{Re}\,\boldsymbol{H}_r)$$
$$= \left[\frac{\beta}{\omega\mu}\boldsymbol{z}_0\times\boldsymbol{e}(\varrho)\cos\beta z + \frac{\sin\beta z}{\omega\mu}\nabla\times\boldsymbol{e}(\varrho)\right][V_1(z) - V_2(z)],$$
$$(\mathrm{Im}\,\boldsymbol{H}_i + \mathrm{Im}\,\boldsymbol{H}_r)$$
$$= \left[\frac{-\beta}{\omega\mu}\boldsymbol{z}_0\times\boldsymbol{e}(\varrho)\sin\beta z + \frac{\cos\beta z}{\omega\mu}\nabla\times\boldsymbol{e}(\varrho)\right][V_1(z) + V_2(z)],$$
$$(\mathrm{Re}\,\boldsymbol{E}_i + \mathrm{Re}\,\boldsymbol{E}_r) = [V_1(z) + V_2(z)]\boldsymbol{e}(\varrho)\cos\beta z,$$
$$(\mathrm{Im}\,\boldsymbol{E}_i + \mathrm{Im}\,\boldsymbol{E}_r) = [-V_1(z) + V_2(z)]\boldsymbol{e}(\varrho)\sin\beta z. \tag{11}$$

It is seen that when these expressions are substituted into (9) the cross terms typified by

$$\left[\frac{\beta}{\omega\mu}\boldsymbol{z}_0\times\boldsymbol{e}(\varrho)\cos\beta z\right]\cdot\left[\frac{\sin\beta z}{\omega\mu}\nabla\times\boldsymbol{e}(\varrho)\right]$$

vanish and one obtains

$$D(z) = (V_1^2 + V_2^2)\frac{1}{(\omega\mu)^2}\oint[\beta^2|\boldsymbol{e}(\varrho)|^2 + |\nabla\times\boldsymbol{e}(\varrho)|^2]r\,dl$$
$$+ 2V_1V_2\cos 2\beta z\frac{1}{(\omega\mu)^2}\oint[|\nabla\times\boldsymbol{e}(\varrho)|^2 - \beta^2|\boldsymbol{e}(\varrho)|^2]r\,dl$$
$$+ (V_1^2 + V_2^2 + 2V_1V_2\cos 2\beta z)\sigma\int_s|\boldsymbol{e}(\varrho)|^2 ds, \tag{12}$$

where, for convenience, the z dependence of V_1 and V_2 is no longer explicitly shown. The last integral is equal to unity by definition. The other two integrals are constants of the problem, *i.e.*, they have no z dependence. One can therefore define

$$G_1 \equiv \frac{r}{Z_0^2}\oint\left[\frac{1}{\beta^2}\,|\nabla\times e(\varrho)|^2 + |e(\varrho)|^2\right]dl \quad (13)$$

$$G_2 \equiv \frac{r}{Z_0^2}\oint\left[\frac{1}{\beta^2}\,|\nabla\times e(\varrho)|^2 - |e(\varrho)|^2\right]dl, \quad (14)$$

where $Z_0\beta$ has been substituted for $\omega\mu$. In terms of G_1 and G_2, (12) gives the dissipation per unit length as

$$D(z) = (V_1^2 + V_2^2)(G_1+\sigma) + 2V_1V_2(G_2+\sigma)\cos 2\beta z. \quad (15)$$

Nothing has yet been said about the (exponentially damped) z dependence of V_1 and V_2,

$$V_1(z) = V_1(0)e^{-\alpha z}, \qquad V_2(z) = V_2(0)e^{\alpha z},$$

which, since the damping is small, may usually be neglected. One can also express V_2 as $|\Gamma|\,V_1$, where Γ is the reflection coefficient of the load. Eq. (15) may therefore be rewritten such that the dependence of $D(z)$ on the termination becomes explicit.

$$D(z) = V_1^2[(1+|\Gamma|^2)(G_1+\sigma) + 2|\Gamma|(G_2+\sigma)\cos 2\beta z]. \quad (16)$$

It must be remembered that z is measured from a voltage maximum, *not* from the termination.

Eq. (16) shows that there is a periodic term in the dissipation expression which depends on the load reflection coefficient magnitude. For a unity reflection coefficient the variation in the dissipation is greatest. In the absence of a reflected wave, dissipation is a simple exponential function of z. The constants G_1 and G_2 are in general unequal; G_2 may even be zero. This point appears to have been overlooked even by authors who were aware of the basic phenomenon.[5]

TABLE I
VALUES OF G_1 AND G_2 FOR TE_{10} MODE IN X-BAND WAVEGUIDE

Frequency	G_1	G_2
8 kmc	2.4×10^{-4}	1.08×10^{-4}
10 kmc	1.12×10^{-4}	-5.8×10^{-7}
12 kmc	5.4×10^{-5}	-1.56×10^{-5}

Table I gives typical values of G_1 and G_2 calculated for an empty brass X-band waveguide operated in the TE_{10} mode. It is of interest to note that the sign of G_2 is positive at 8 kmc and negative at 10 kmc. Near 10 kmc G_2, and consequently the periodic part of the dissipation, are vanishingly small. The physical explanation for this is that the transverse and longitudinal magnetic field components of the traveling waves do not add in phase. The transverse components add at what are conventionally called current maxima and the longitudinal components at the voltage maxima. In the 10-kmc region, the contributions of the two components to the dissipation are about equal, so that G_2 is small and dissipation occurs relatively uniformly along the waveguide. Above 10 kmc the transverse component is predominant, G_2 is negative, and maximum dissipation occurs at the current maxima. Below 10 kmc the longitudinal component is mainly responsible for the power loss so that G_2 is positive and the maximum dissipation coincides with the voltage maxima.

[5] H. E. King, "Rectangular waveguide theoretical CW average power rating," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-9, pp. 349–357; July, 1961. See especially p. 356. Eq. (21) is inapplicable in the TE mode case.

TM MODE

The analysis in the TM case is carried out in the same way as that in the TE case and only the results need be stated here.

$$D(z) = (V_1^2 + V_2^2)(G_3+\sigma) + 2V_1V_2(G_4+\sigma)\cos 2\beta z, \quad (17)$$

where

$$G_3 = -G_4 \equiv \frac{\sigma}{\beta^2}\int_s |\nabla\times h(\varrho)|^2\,ds + \frac{r}{Z_0^2}\oint |h(\varrho)|^2 dl. \quad (18)$$

Here $h(\varrho)$ is the normalized magnetic field mode function.[4] Eq. (17) may also be expressed in terms of the reflection coefficient of the termination. When this has been done, the TE and TM cases may be summarized in the equation

$$D(z) = V_1^2[(1+|\Gamma|^2)(G_{\substack{1\\3}}+\sigma) + 2|\Gamma|(G_{\substack{2\\4}}+\sigma)\cos 2\beta z]_{\mathrm{TM}}^{\mathrm{TE}}. \quad (19)$$

TRANSMISSION LINE APPROACH

The dissipation per unit length can also be derived using the solution of the transmission line equations as a starting point. The result is equally applicable to TE, TM and TEM mode transmission lines.

It is well known that for a transmission line characterized by Z, a series impedance per unit length, and Y, a shunt admittance per unit length, the voltage and current solutions may be written

$$V = Ae^{-\gamma z} + Be^{\gamma z} \quad (20)$$

$$I = Y_0[Ae^{-\gamma z} - Be^{\gamma z}] \quad (21)$$

where

$$\gamma = \alpha + j\beta = \sqrt{YZ} \quad (22)$$

$$Y_0 = \sqrt{Y/Z}. \quad (23)$$

To a very good approximation the choice of $z=0$ at a voltage maximum assures that both A and B be real and positive.

The (complex) power flow toward the load at any point z is

$$P_c(z) = V(z)I^*(z), \quad (24)$$

and the (complex) dissipation which is the decrease in $P_c(z)$ with z is

$$-\frac{dP_c}{dz} = -\frac{d(VI^*)}{dz}. \quad (25)$$

Substitution of (20) and (21) into (25) yields

$$-\frac{dP_c}{dz} = 2\alpha Y_0^*[A^2e^{-2\alpha z} + B^2e^{2\alpha z}] - j4\beta Y_0^* AB \cos 2\beta z. \quad (26)$$

The quantity of interest here is the true dissipation, the real part of (26). Hence,

$$D(z) = 2\alpha \operatorname{Re}(Y_0)[A^2e^{-2\alpha z} + B^2e^{2\alpha z}] - 4\beta \operatorname{Im}(Y_0)AB \cos 2\beta z. \quad (27)$$

It is observed here that a periodic term is present in a case where there can be no question as to the damped exponential nature of the traveling waves. This should remove any intuitive doubts about the compatibility of damped exponential solutions with periodic variations in the dissipation. Eq. (27) shows that the periodic part of the dissipation expression is associated with the imaginary part of the characteristic admittance. It can be shown that *β Im (Y_0) is usually of the same order of magnitude as α Re (Y_0) and therefore it cannot be neglected.*

Application

Total dissipation D_t is obtained by integrating the appropriate dissipation per unit length expression, (19) or (27), over the length of the waveguide, *i.e.*,

$$D_t = \int_{z_1}^{z_2} D(z)dz. \quad (28)$$

It is observed that when (z_2-z_1) equals an integral number of half wavelengths, the cosine term does not contribute to the integral. In that case the total dissipation is accurately given when it is assumed that the power attenuation associated with each wave is $\exp(-2\alpha|z_2-z_1|)$. It is also seen that when the waveguide is many wavelengths long, the error introduced by neglecting the cosine term is small.

The periodic part of the dissipation expression may be neglected when one considers high Q transmission or reflection cavities which are very nearly an integral number of half wavelengths long at resonance, and when one calculates the dissipation in long waveguides. On the other hand, reflection cavities embodying reactive terminations other than short circuits, cavities partially filled with dielectric,[6,7] etc., are not necessarily $n\lambda/2$ in length at resonance. Therefore, in calculating the Q of such cavities, the expressions for dissipation given here should be used.

The periodic concentration of the dissipation could also prove troublesome in attenuators. A case in point is the metallized glass coaxial attenuator in which the inner conductor is a fragile metal film. Attenuation is entirely due to I^2R loss, for which it may be shown that $\beta \operatorname{Im}(Y_0) = \alpha \operatorname{Re}(Y_0)$ [see (27)]. Therefore, in the extreme case of a reactively terminated attenuator, the peak dissipation as a function of distance is twice the average dissipation. The same situation, though to a lesser degree, can also result from internal standing waves in an otherwise matched attenuator.

Acknowledgment

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